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Baseband Linearity and Equalization in Fiber Optic Digital Communication Systems

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If a sequence of digitally on-off modulated optical pulses is injected into a dielectric waveguide, these pulses may begin to overlap after a sufficient distance of propagation because of material dispersion and/or group delay spreading. In general, the pulses will not add linearly in power, which can complicate the problem of equalization of the square-law (power) detected overlapping output pulses at baseband. This paper illustrates important situations in which the guide may be treated as "pseudo-linear" in power, meaning that the detected guide output pulses appear to add linearly.

I. INTRODUCTION

If a single pulse of optical energy propagates along a dielectric waveguide, pulse broadening can occur for one or more of the following reasons: material dispersion, individual mode waveguide dispersion, or differences in the group delays of different guide modes. In addition, the pulse shape may become only statistically defined because of random mode coupling and/or statistical fluctuations of the optical source.

If a sequence of digitally on-off modulated pulses is injected into a dielectric waveguide, those pulses may begin to overlap after a sufficient distance of propagation. In general, the optical powers in the pulses will not add in a linear manner.† On the other hand, as will be shown below, the guide may be pseudo-linear in power. That is, for the purpose of processing the power received at the output end of the

† The fiber is a medium which is linear *in E field propagation* (from Maxwell's equations). It is usually excited by a power-modulated source, and its output field is detected by a square-law (power sensitive) device. Even if the input pulses are separate, the response of the square-law detector will in general contain cross terms resulting from the overlap in the output pulses.

guide—in order to make decisions as to whether or not each pulse is on or off—we may be able to treat the individual overlapping output pulses as if they added linearly.

If the output power pulses could be considered to add linearly, then, after detection, the resulting current pulses could be separated with a linear equalizer provided the shapes of the pulses are well defined and identical from pulse to pulse. In this paper we consider a number of interesting cases in which the guide can be treated as if it were linear in power (output pulses add linearly) and in which the pulses at the output assume a well-defined shape in spite of random mode coupling and/or source fluctuations.

II. CASE I: MULTIMODE GUIDE, MODE-LOCKED SOURCE, NO MODE COUPLING

The easiest case to visualize in which the guide appears linear in power with overlapping output pulses is that of a multimode guide propagating pulses derived from a mode-locked laser operating in a single spatial mode. It is assumed that there is no mode conversion and that group delay differences among the modes dominate pulse spreading.

Even though the laser puts out a well-defined spatial mode, it may be very difficult to match this to a given fiber mode so that only one fiber mode is excited. We assume that a number of fiber modes are excited by the pulses from the mode-locked laser. The assumption of a mode-locked laser implies that the optical bandwidth being used is small so that material and waveguide dispersion can be neglected, and, in addition, no random fluctuations are present from beating of unlocked source modes. The sequence of nonoverlapping pulses from the laser exciting the guide input will produce identical sequences of nonoverlapping pulses in *each mode* at the guide output. However, the sequences at the output in the various modes will have relative time delays because of the differences in the group delay per unit length associated with the various modes (see Fig. 1). When the fiber output falls upon a detector, the current produced (neglecting shot noise) is proportional to the sum of the powers in all the modes. The sum of the powers in all the modes resulting from a *single input pulse* is shown in Fig. 1. Since the output pulses in a single mode do not overlap and since the detector linearly adds the powers of the various modes, the total detected current will be a sum of pulses modulated on and off, each of which looks like the response to a single input pulse. Thus, the output power produces a detected current which is a filtered version of

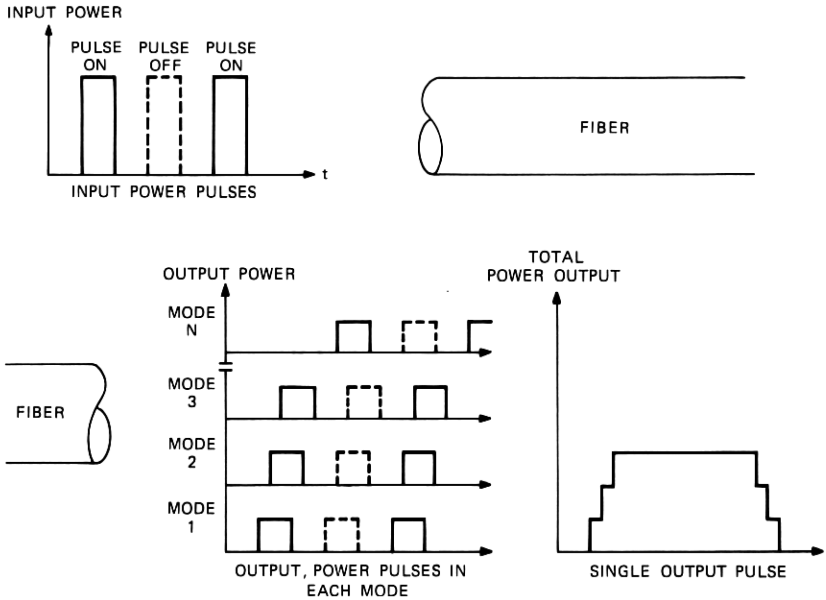

Fig. 1—Multimode propagation without coupling.

the input power. That is, the sequence of nonoverlapping input pulses produces a detected current at the output, which is a sequence of overlapping pulses that add linearly. For the purposes of processing this current, the guide can therefore be considered linear in power or linear at baseband.

III. CASE II: INCOHERENT SOURCE, MULTIMODE GUIDE, NO MODE COUPLING

In this example, we assume that a pulse modulated incoherent source excites one or more modes of a multimode waveguide with no mode coupling. We show that the received output power *in each mode* can be treated as a linearly filtered version of the input power, i.e., that the sequence of nonoverlapping input pulses produces a sequence of output pulses in each mode which add up linearly, whether they overlap (because of material dispersion) or not. Since the detector produces a current which is proportional to the sum of the powers in each mode, the total current will also consist of a sequence of pulses which add linearly, whether they overlap or not. We show that, in order for this effective linearity in power to be valid, it is necessary that the source bandwidth be sufficiently large compared to the reciprocal of the input pulse duration. How big the ratio of these two quantities must be depends upon how much overlap there is in the output pulses and therefore upon how much equalization is required to separate the pulses at baseband.

We can model the complex amplitude of a given spatial mode at the guide input as follows:

$$\epsilon_{\text{in}}(t) = \sqrt{m(t)}\,c(t). \tag{1}$$

In the above example, $(m(t))^{\frac{1}{2}}$ represents the modulation and $c(t)$ is a complex Gaussian random process which represents the incoherent carrier. By definition of an incoherent carrier, we have

$$\langle c(t)c(t+u)\rangle = 0, \qquad \langle c(t)c^*(t+u)\rangle = R_c(u). \tag{2}$$

The Fourier transform of $R_c(u)$ is what is called the incoherent source spectrum, shifted to baseband.

The input complex amplitude of (1) produces an output from the guide in the corresponding mode having the following complex amplitude

$$\epsilon_{\text{out}}(t) = \int \epsilon_{\text{in}}(t')h_g(t-t')dt'. \tag{3}$$

In eq. (3), $h_g(u)$ is the guide bandpass impulse response for the mode under consideration. Equation (3) follows from the fact that the guide is linear in voltage.

The average power at the guide input in the given mode is (averaging over the fluctuations in the incoherent carrier)

$$\langle p_{\text{in}}(t)\rangle \triangleq \langle \epsilon_{\text{in}}(t)\epsilon_{\text{in}}^*(t)\rangle = m(t)R_c(0) = R_c(0)\left[\sum a_k h_{p_{\text{in}}}(t-kT)\right]. \tag{4}$$

In (4), the modulation $m(t)$ is a sequence of nonoverlapping pulses (modulated on or off) where $h_{p_{\text{in}}}(t)$ is the pulse shape, a_k assumes the value zero or one for each k, and T is the pulse spacing. The average power at the guide output is

$$\langle p_{\text{out}}(t)\rangle = \left\langle \iint \epsilon_{\text{in}}(t')h_g(t-t')\epsilon_{\text{in}}^*(t'')h_g^*(t-t'')dt'dt''\right\rangle. \tag{5}$$

It is reasonable to assume that the following approximation holds:

$$\langle \epsilon_{\text{in}}(t')\epsilon_{\text{in}}^*(t'')\rangle = \langle \sqrt{m(t')}c(t')\sqrt{m(t'')}c^*(t'')\rangle = \sqrt{m(t')}\sqrt{m(t'')}\,R_c(t'-t'') \approx m(t')R_c(t'-t''). \tag{6}$$

This approximation is valid since the coherence time of a typical incoherent source such as a GaAs LED is of the order of 10^{-13} seconds, while modulation pulse widths of interest here exceed 10^{-9} seconds.

Substituting (6) into (5) we obtain

$$\langle p_{\text{out}}(t)\rangle = \sum a_k h_{p_{\text{out}}}(t - kT), \tag{7}$$

where

$$h_{p_{\text{out}}}(t) = \int h_{p_{\text{in}}}(t') \left[\int R_c(t' - t'') h_g(t - t') h_g^*(t - t'') dt'' \right] dt'.$$

Thus, the average output power is a linearly filtered version of the average input power. That is, the average output power consists of a sequence of pulses which add linearly even if they overlap.

Thus far, we have considered the average output power, averaging over the fluctuations in the incoherent source power output. We next consider the effect of those fluctuations on the equalized-detected current. We shall show that these source fluctuations will produce negligible deviations in the equalized-detected current from its mean provided that the source bandwidth is sufficiently large.

We can write the power at the guide output in the mode under consideration as the sum of the average power of eq. (7) and a deviation from this average $b(t)$:

$$p_{\text{out}}(t) = \langle p_{\text{out}}(t)\rangle + b(t). \tag{8}$$

If this power falls upon a detector, it produces a current which is proportional to $p_{\text{out}}(t)$ (neglecting shot noise). This current will pass through a filter which performs an equalization function and/or band-limiting. The baseband filter output voltage will therefore be

$$v_{\text{out}}(t) = z \int p_{\text{out}}(t') h_b(t - t') dt', \tag{9}$$

where $h_b(t - t')$ is the baseband filter impulse response and z is an arbitrary proportionality constant.

The mean baseband output voltage is given by

$$\langle v_{\text{out}}(t)\rangle = z \int \langle p_{\text{out}}(t')\rangle h_b(t - t') dt'. \tag{10}$$

The mean square deviation of the baseband voltage from its mean is given by

$$\langle v_{\text{out}}^2(t)\rangle - [\langle v_{\text{out}}(t)\rangle]^2 = \sigma_v^2(t) \triangleq z^2 \int \langle p_{\text{out}}(t') p_{\text{out}}(t'')\rangle \times h_b(t - t') h_b(t - t'') dt' dt'' - [\langle v_{\text{out}}(t)\rangle]^2. \tag{11}$$

Thus, to calculate the ratio of the mean voltage to the rms deviation in order to determine whether or not the deviations are negligible, we need the correlation function of the power $p_{\text{out}}(t)$.

In order to calculate this correlation function, we must recall that $c(t)$ defined in (1) is a complex Gaussian random process and satisfies [in addition to (2)] the following

$$\langle c(t)c^*(t')c(t'')c^*(t''')\rangle = R_c(t-t')R_c(t''-t''') + R_c(t-t''')R_c(t''-t'). \quad (12a)$$

Using (1), (3), (4), and (12a) we obtain

$$\langle p_{\text{out}}(t)p_{\text{out}}(t')\rangle = \langle p_{\text{out}}(t)\rangle\langle p_{\text{out}}(t')\rangle + \left|\int m(\alpha)R_c(\alpha-\beta)h_g(t-\alpha)h_g^*(t'-\beta)d\alpha d\beta\right|^2. \quad (12b)$$

To obtain some numerical results, we assume that the input pulses $h_{p_{\text{in}}}(t)$ defined in (4) are Gaussian in shape and that the guide mode impulse response corresponds to that of a dispersive medium having a group delay τ_0 at the optical source center frequency and a dispersion γ^2 within the optical band of the source. Further, we shall assume that the source spectrum is Gaussian in shape. That is, we shall assume the following:

$$h_{p_{\text{in}}}(t) = \exp(-t^2/2\sigma^2), \quad (2.36\sigma = \text{input pulse width between } \tfrac{1}{2} \text{ points}) \quad (13a)$$

$$\mathfrak{F}\{h_g(t)\} \triangleq H_g(\omega) = \exp\left(-j\frac{\gamma^2\omega^2}{2}\right)\exp(-j\tau_o\omega)\exp(-\omega^2 B_g^{-2}/2) \quad (13b)$$

$$\begin{aligned} h_g(t) &= \frac{1}{\sqrt{2\pi(j\gamma^2+B_g^{-2}}}\exp[-(t-\tau_o)^2/2(j\gamma^2+B_g^{-2})] \\ R_c(u) &= \exp(-u^2B_s^2/2), \end{aligned} \quad (13c)$$

where

$$\mathfrak{F}\{\ \} = \text{Fourier transform.}$$

In eq. (13), we have already assumed that the source bandwidth B_s is much greater than the reciprocal of the input pulse width σ. We also shall assume that the guide bandwidth B_g is much larger than the source bandwidth B_s. Since τ_o represents an absolute propagation delay from input to output, we shall neglect it as irrelevant to the problem at hand. Therefore, the only significant parameter in the guide impulse response $h_g(t)$ is the dispersion γ^2.

If we insert the particular functions of (13) into (7) and (12b) using (4), we obtain

$$\langle p_{\text{out}}(t)\rangle = z_2 \sum a_k \exp\{-\tfrac{1}{2}(t-kT)^2/[\gamma^4B_s^2+\sigma^2]\} \quad (14a)$$

(i.e., a sequence of Gaussian-shaped on-off modulated pulses having width $\sqrt{\gamma^4 B_s^2 + \sigma^2}$), where z_2 = an arbitrary proportionality constant which we shall henceforth set to unity.

$$\langle p_{\text{out}}(t) p_{\text{out}}(t') \rangle = [\langle p_{\text{out}}(t) \rangle]^2 \left[1 + \exp\left\{ - \frac{(t-t')^2 B_s^2}{\left[\frac{\gamma^4 B_s^2}{\sigma^2} + 1 \right]} \right\} \right]. \tag{14b}$$

Thus, $h_{p_{\text{out}}}(t)$ of (8) is in this case the Gaussian-shaped pulse in (14a). Looking at eqs. (13a) and (14a), we see that $\gamma^4 B_s^2/\sigma^2 \ll 1$ implies that little pulse broadening has occurred in propagation; $\gamma^4 B_s^2/\sigma^2 \gg 1$ implies that considerable pulse broadening has occurred in propagation. Now using (14) in (11), we obtain

$$\frac{[\langle v_{\text{out}}(t) \rangle]^2}{\sigma_v^2(t)} = \frac{\left(\int \langle p_{\text{out}}(t') \rangle h_b(t-t') dt' \right)^2}{\int [p_{\text{out}}(t')]^2 h_b(t-t') h_b(t-t'') e^{-(t'-t'')^2 u^2/2}} dt' dt'', \tag{15}$$

where

$$u^2 = 2B_s^2 [\gamma^4 B_s^2/\sigma^2 + 1]^{-1}.$$

Recall that $\langle v_{\text{out}}(t) \rangle$ is the average baseband (detected and equalized) voltage produced by the power output in the mode under consideration, and that it consists of a sum of on-off modulated pulses given in (14a). In addition, $\sigma_v^2(t)$ is the mean squared deviation of this baseband voltage from its mean. Thus, (15) is effectively a signal-to-noise ratio. For a typical broadband source and an equalizer having a bandwidth comparable to 1/(pulse spacing-T), we can treat the Gaussian term of the integrand in the denominator of (15) as a delta function having area $\sqrt{2\pi}/u$. Then (15) becomes

$$\frac{[\langle v_{\text{out}}(t) \rangle]^2}{\sigma_v^2(t)} = \frac{\left[\int \langle p_{\text{out}}(t') \rangle h_b(t-t') dt' \right]^2}{\frac{\sqrt{2\pi}}{u} \left[\int \langle p_{\text{out}}(t') \rangle^2 h_b^2(t-t') dt' \right]}. \tag{16}$$

We can evaluate (16) for particular equalizers, $h_b(t)$, and particular output power pulse widths $(\gamma^4 B_s^2 + \sigma^2)$. We recognize from (16) in general that the equivalent noise $b(t)$ of (8) which must be added to the average output power is a signal-dependent noise with correlation function

$$\langle b(t) b(t') \rangle \cong \frac{\sqrt{2\pi}}{u} [\langle p_{\text{out}}(t) \rangle]^2 \delta(t-t'), \tag{17}$$

where

$$u^2 = 2B_s^2/[\gamma^4 B_s^2/\sigma^2 + 1].$$

If the power output pulses $h_{p_{\rm out}}(t)$ overlap significantly, then this signal-dependent noise will become stationary when all the pulses are on, which should simplify the calculation of (16). At the other extreme, if the output pulses $h_{p_{\rm out}}(t)$ do not overlap (i.e., $\gamma^4 B_s^2 + \sigma^2 \ll T^2$) and if the baseband equalizer $h_b(t)$ is taken to be a matched filter (matched in shape to the output pulses), then the signal-to-noise ratio is given by

$$\frac{[\langle v_{\rm out}(t)\rangle]^2}{\sigma_v^2(t)} = \sqrt{2}\sigma B_s \tag{18}$$

[if $T^2 \gg \gamma^4 B_s^2 + \sigma^2$, $h_b(t) = h_{p_{\rm out}}(t)$].

In general, for a given desired equalized pulse shape, the equivalent noise will be negligible if the product of the optical source bandwidth B_s and the input pulse width σ is sufficiently large. For practical cases of interest, this product is on the order of 10^4 to 10^5. For reasonable amounts of equalization consistent with other noise considerations (shot noise, thermal noise, etc.), we can treat the guide as being linear in power even if the pulses overlap, i.e., we can neglect the equivalent noise $b(t)$.

When more than one mode is present, we simply add the individual mode output powers, since the detector current is proportional to the sum of the powers in all the modes. If the optical source is spatially incoherent, the equivalent noises $b(t)$ in each mode may be uncorrelated.† In that case, the requirements upon the product of the source bandwidth and the input pulse width are less stringent. This is particularly true if the pulse spreading resulting from dispersion dominates the spreading resulting from the differences in group delay among the various modes.

A simple interpretation which may prove useful follows. It was easy to obtain (7), which showed that the guide was linear in power if we averaged out the source fluctuations. Since the source is very broadband, we can think of it as a sum of independently fluctuating sources separated by a frequency spacing equal to the bandwidth of the modulation pulses at the input. Thus, the output power is the sum of the fluctuating powers associated with each equivalent independently fluctuating optical source. The average output powers associated with these equivalent sources add systematically, while the indepen-

† That is, if the optical source is close to the fiber, each fiber mode effectively sees an independent carrier, and therefore we obtain averaging of the fluctuations in these carriers when the mode powers at the output of the guide are added.

dent fluctuations about the average add at random. Thus, for a sufficiently large number of equivalent sources—corresponding to a large product of optical source bandwidth and input pulse width—the total fluctuations become small compared to the average power. In effect, one has a frequency diversity system.

IV. CASE III: MODE-LOCKED COHERENT SOURCE, MULTIMODE GUIDE, MODE COUPLING

In this example, we consider a spatially coherent mode-locked laser source and a multimode guide. Unlike example I, we assume considerable mode coupling. Once again, the rationale of using a multimode guide with a single mode source may be the inability to stably match the source to a single-mode guide. Before proceeding, we must model the transmission properties of a multimode guide with random coupling.

Very little is known about the complete statistical properties of the guide under consideration here. Any particular guide, which is linear in voltage (field), can be characterized as having a set of modes associated with an ideal guide having no geometry perturbations (which are the source of coupling). The input and output complex envelopes of the corresponding input and output optical fields can be expanded using the orthogonal guided modes, which together with the continuum of radiating modes form a complete orthonormal series. We can relate the complex amplitudes in each input and output mode by a matrix impulse response. That is, calling the complex amplitude in mode k at the input $\epsilon_{k_{\text{in}}}(t)$, and the complex amplitude in mode j at the output $\epsilon_{j_{\text{out}}}(t)$, we have

$$\epsilon_{j_{\text{out}}}(t) = \sum_k \int \epsilon_{k_{\text{in}}}(t')h_{jk}(t - t')dt', \tag{19}$$

where $h_{jk}(t - t')$ is the bandpass impulse response from input mode k to output mode j.

In order to proceed in the analysis to follow, we need at least the fourth-order joint statistics of the random processes $h_{jk}(t)$. (The reader is cautioned that the term *random process* refers to the fact that the actual $h_{jk}(t)$ for each j and k will be different for different guides because of the random mode coupling. For any particular guide which does not change its physical parameters in time, $h_{jk}(t)$ is a fixed but *a priori* unpredictable function of time. All averaging and references to statistical properties refer to ensembles of guides whose gross physical properties are alike.)

As mentioned above, little is known about the statistics of the $h_{jk}(t)$. Rowe and Young,[1] Personick,[2] and Marcuse[3] have shown in various analyses that, for a particular j and k, the Fourier transform of $h_{jk}(t)$, $H_{jk}(\omega)$ can be considered a stationary random process under various restricted conditions which include the assumption that the optical bandwidth being used is not too large. That is, one may argue under restricted conditions that

$$\langle H_{jk}(\omega)H_{jk}^*(\omega+\sigma)\rangle = S_{jk}(\sigma) \tag{20}$$

(where the averaging is over an ensemble of guides having identical gross properties).

In another analysis, Marcuse[4] has shown that, for a particular j and k and for a sufficiently long guide (so that enough mode coupling has taken place), one has

$$\langle |H_{jk}(\omega)|^4\rangle \approx 2\langle |H_{jk}(\omega)|^2\rangle^2.$$

This last result is consistent with (but certainly not a sufficient condition for) the possibility that $H_{jk}(\omega)$ is a complex Gaussian random process.

Based on this admittedly scanty evidence which should certainly be explored in more depth, we shall *assume* that the Fourier transforms $H_{jk}(\omega)$ of the $h_{jk}(t)$ satisfy the following conditions which would be satisfied if the $H_{jk}(\omega)$ were joint complex Gaussian random processes

$$\begin{gathered}\langle H_{jk}(\omega)H_{lm}(\omega+\sigma)\rangle = 0, \qquad \langle H_{jk}(\omega)H_{lm}^*(\omega+\sigma)\rangle = S_{jklm}(\sigma)\\ \langle H_{jk}(\omega)H_{lm}^*(\omega+\sigma)H_{no}(\omega+\sigma')H_{pq}^*(\omega+\sigma'')\rangle \\ = S_{jklm}(\sigma)S_{nopq}(\sigma''-\sigma') + S_{jkpq}(\sigma'')S_{nolm}(\sigma-\sigma').\end{gathered} \tag{21}$$

It is hoped that the results which we shall next derive will be qualitatively valid for actual multimode guides with random mode coupling.

The guide input optical field complex amplitude is given by

$$\epsilon_{\text{in}}(t) = \epsilon_p(\rho)\epsilon_\tau(t), \tag{22}$$

where $\epsilon_p(\rho)$ represents the spatial variation of the field over the guide input plane and $\epsilon_\tau(t)$ represents the time variation of the field and includes the modulation. We shall assume that the modulation consists of a sum of nonoverlapping on-off modulated pulses with spacing T

$$\begin{aligned}\epsilon_\tau(t) &= \sum a_k h_{\text{in}}(t-kT),\\ a_k &= 0 \quad \text{or} \quad 1 \quad \text{for each } k.\end{aligned} \tag{23}$$

We can expand the input field in the guide modes as follows:

$$\epsilon_p(\rho) = \sum \epsilon_k\phi_k(\rho) + \text{unguided remainder}, \tag{24}$$

where $\phi_k(\rho)$ is guided mode k.

From (20) and (24), we obtain the complex amplitude in mode j at the guide output in terms of the input complex envelope and an impulse response

$$\begin{aligned}\epsilon_{j_{\text{out}}}(t) &= \sum_k \int \epsilon_k \epsilon_\tau(t') h_{jk}(t-t')dt' \\ &= \int \epsilon_\tau(t') h_j(t-t')dt', \qquad \text{where } h_j(t) \triangleq \sum_k \epsilon_k h_{jk}(t). \end{aligned} \tag{25}$$

Since $h_j(t)$ is a weighted sum of individual responses $h_{jk}(t)$, it follows from (21) and the linearity properties of the Fourier transform that the transform $H_j(\omega)$ of $h_j(t)$ must also satisfy

$$\begin{gathered}\langle H_j(\omega)H_j^*(\omega+\sigma)\rangle = S_j(\sigma), \qquad \langle H_j(\omega)H_j(\omega+\sigma)\rangle = 0 \\ \langle H_j(\omega)H_j^*(\omega+\sigma)H_j(\omega+\sigma')H_j^*(\omega+\sigma'')\rangle \\ = S_j(\sigma)S_j(\sigma''-\sigma') + S_j(\sigma'')S_j(\sigma-\sigma'). \end{gathered} \tag{26}$$

We next show that the guide may be considered under restricted circumstances to be linear in power with a well-defined output pulse shape even though the output pulses overlap and even though there is unpredictable mode coupling.

First, we can write down the power at the guide input and at the guide output in mode j:

$$p_{\text{in}}(t) = \left[\int |\epsilon_p(\rho)|^2 d_\rho^2\right] \sum a_k |h_{\text{in}}(t-kT)|^2 \tag{27}$$

(since $h_{\text{in}}(t)h_{\text{in}}^*(t-kT) = 0$ for $k \neq 0$, and $a_k = 0$ or 1).

$$p_{j_{\text{out}}}(t) = \int \epsilon_\tau(t')\epsilon_\tau^*(t'')h_j(t-t')h_j^*(t-t'')dt'dt''.$$

Next, we can ensemble average $p_{j_{\text{out}}}(t)$ over the ensemble of similar guides to find the average power response.

In Reference 2, it is shown that (26) implies that

$$\langle h_j(t-t')h_j^*(t-t'')\rangle = s_j(t-t')\delta(t'-t''), \tag{28}$$

where $s_j(t) = \mathfrak{F}^{-1}\{S_j(\omega)\}$ (inverse Fourier transform).

Using (28) and (27) we obtain

$$\langle p_{j_{\text{out}}}(t)\rangle = \sum a_k h_{j_{\text{out}}}(t-kT),$$

where

$$h_{j_{\text{out}}}(t) = \int |h_{\text{in}}(t')|^2 s_j(t-t')dt'. \tag{29}$$

We thus see that the input power consists of a sum of on-off modulated overlapping pulses, and the average output power (averaged over an ensemble of guides with identical gross physical properties) consists of a similar sum of on-off modulated pulses which in general may overlap. Thus, in this ensemble-averaged sense, the guide is linear in power with impulse response $s_j(t)$. We must next investigate how an individual guide in the ensemble can deviate from this average and under what conditions these deviations can be neglected for communications purposes.

In order to study these deviations we must consider them in the context of a detector followed by an equalizing (or simply band-limiting) filter. Since the detector produces a current which is proportional to the linear sum of the powers in each guide mode at the output (neglecting shot noise), we consider the response to one mode only for the moment. The voltage at the equalizer output is related to the output power in mode j as follows:

$$v_{\text{out}}(t) = z \int p_{j_{\text{out}}}(t') h_{\text{det.-filt.}}(t - t') dt', \tag{30}$$

where $h_{\text{det.-filt.}}(t)$ is the detector-filter impulse response and z is an arbitrary constant.

The average (over an ensemble of guides) voltage produced is a sum of on-off modulated pulses, since we have already shown that the average output power in mode j is a sum of on-off modulated pulses

$$\langle v_{\text{out}}(t) \rangle = \sum a_k h_{v_{\text{out}}}(t - kT), \tag{31}$$

where

$$h_{v_{\text{out}}}(t) = \int h_{j_{\text{out}}}(t') h_{\text{det.-filt.}}(t - t') dt'.$$

The mean squared deviation from this average voltage is given as follows:

$$\begin{aligned} \sigma_v^2(t) &\triangleq \langle v_{\text{out}}^2(t) \rangle - \langle v_{\text{out}}(t) \rangle^2 \\ &= z^2 \iint \langle p_{j_{\text{out}}}(t') p_{j_{\text{out}}}(t'') \rangle h_{\text{det.-filt.}}(t - t') \\ &\qquad \times h_{\text{det.-filt.}}(t - t'') dt' dt'' - \langle v_{\text{out}}(t) \rangle^2. \end{aligned} \tag{32}$$

To calculate the mean squared deviation, we need the correlation function of the output power in mode j. From (27) we obtain

$$\begin{aligned} \langle p_{j_{\text{out}}}(\alpha) p_{j_{\text{out}}}(\beta) \rangle &= \iiiint [\epsilon_\tau(t') \epsilon_\tau^*(t'') \epsilon_\tau(t''') \epsilon_\tau^*(t'''')] \\ &\times \langle h_j(\alpha - t') h_j^*(\alpha - t'') h_j(\beta - t''') h_j^*(\beta - t'''') \rangle dt' dt'' dt''' dt''''. \end{aligned} \tag{33}$$

In order to obtain numerical results, we must make some assumptions to facilitate the products and convolutions of (33). We assume that the guide power impulse response $s_j(t)$ for mode j is Gaussian in shape. It has been shown[1,2] that this should be the case under restricted conditions for long guides. We assume that the input pulses, $h_{\text{in}}(t)$, are Gaussian in shape with a width less than $\frac{1}{3}$ the pulse spacing, T, so that the previous assumption that they do not overlap is not violated for practical purposes. That is, we assume

$$\begin{aligned} h_{\text{in}}(t) &= e^{-t^2/2\sigma^2}, \qquad 3\sigma < T = \text{pulse spacing} \\ s_j(t) &= e^{-t^2/2\gamma^2}, \qquad \gamma > 3\sigma. \end{aligned} \tag{34}$$

From (33), (26), and (34) we obtain

$$\begin{aligned} \langle p_{j_{\text{out}}}(\alpha) \rangle &\cong z \sum a_k \exp - \left\{ (\alpha - kT)^2 \Big/ 2\left[\frac{\sigma^2}{2} + \gamma^2 \right] \right\} \\ \langle p_{j_{\text{out}}}(\alpha) p_{j_{\text{out}}}(\beta) \rangle &\cong \Bigg[z^2 \sum_{kml}\sum\sum a_k a_{k-m} a_l a_{l+m} \\ &\quad \times \exp - \left\{ \frac{(\alpha - kT)^2}{2(\sigma^2/2 + \gamma^2)} \right\} \exp - \left\{ \frac{(\beta - lT)^2}{2(\sigma^2/2 + \gamma^2)} \right\} \\ &\quad \times \exp - \left\{ \frac{[(\alpha - \beta) - mT]^2 \gamma^2}{2(\sigma^2/2 + \gamma^2)\sigma^2} \right\} \Bigg] \\ &\qquad + \langle p_{j_{\text{out}}}(\alpha) \rangle \langle p_{j_{\text{out}}}(\beta) \rangle. \end{aligned} \tag{35}$$

The approximations of (35) become equalities when the width of the average power impulse response γ becomes large compared to the input pulse width, σ, i.e., when there is a lot of pulse spreading in propagation. We shall soon see that this will be the case of interest in this example. Before attempting to use (35) in (32) to evaluate the magnitude of the deviations of a particular guides power from the ensemble average, we can make some simplifications and comments upon (35). Using the assumptions $\gamma \gg \sigma$, we have

$$\begin{aligned} \langle p_{j_{\text{out}}}(\alpha) \rangle &= z \sum a_k \exp \{ - (\alpha - kT)^2 / 2\gamma^2 \} \\ R_b(\alpha, \beta) &\triangleq \langle p_{j_{\text{out}}}(\alpha) p_{j_{\text{out}}}(\beta) \rangle - \langle p_{j_{\text{out}}}(\alpha) \rangle \langle p_{j_{\text{out}}}(\beta) \rangle \\ &= z^2 \sum_{kml}\sum\sum a_k a_{k-m} a_l a_{l+m} e^{[-(\alpha - kT)^2/2\gamma^2]} e^{[-(\beta - lT)^2/2\gamma^2]} \\ &\qquad \times e^{-(\alpha - \beta - mT)^2/2\sigma^2}. \end{aligned} \tag{36}$$

If we assume that σ is small compared to the pulse spacing T and if we assume that the detector-equalizer combination passes frequencies only up to the inverse of the pulse spacing T, then we can make the further approximation

$$R_b(\alpha, \beta) = z^2 \sum_{klm}\sum\sum a_k a_{k-m} a_l a_{l+m} \times [e^{(\alpha-kT)^2/2\gamma^2} e^{-(\beta-lT)^2/2\gamma^2}] \delta(\alpha - \beta - mT)\sqrt{2\pi\sigma^2}. \quad (37)$$

What we have shown so far is that the power in mode j at the guide output can be considered to be of the form

$$p_{j_{\text{out}}}(t) = \langle (p_{j_{\text{out}}}(t) \rangle + b(t), \quad (38)$$

where

$$\langle b(\alpha) b(\beta) \rangle = R_b(\alpha, \beta),$$

where $b(t)$ represents the deviations of the power in a particular guide in mode j from the ensemble average.

Combining (36) and (38) with (32), we obtain the ratio of the (average voltage)^2 at the detector filter output to the mean square deviation from this average voltage

$$\frac{S}{N} \triangleq \frac{\langle v_{\text{out}}(t) \rangle^2}{\sigma_v^2(t)}$$

$$= \frac{\left[\int \sum a_k e^{-(t'-kT)^2/2\gamma^2} h_{\text{det.-filt.}}(t - t') dt' \right]^2}{(\sqrt{2\pi}\sigma) \int \sum\sum\sum a_k a_l a_{k-m} a_{l+m} e^{[-(t'-kT)/2\gamma^2]} e^{[-(t''-lT)/2\gamma^2]} \times \delta(t' - t'' - mT) h_{\text{det.-filt.}}(t - t') h_{\text{det.-filt.}}(t - t'') dt' dt''}. \quad (39)$$

It is clear that, whatever the equalizing filter is, this ratio increases with decreasing σ. Thus, as the input power pulses to the guide become narrow, two things happen: the average output pulse widths become independent of the input pulse width and the deviations of the output power in mode j from the ensemble average become negligible. Exactly how small σ has to be depends upon how much the average output power pulses overlap and how much equalization we are therefore using. In the extreme case of no output pulse overlap, assuming that the equalizer response $h_{\text{det.-filt.}}(t)$ is matched to the average pulse power, we obtain

$$\frac{S}{N} = \frac{\gamma\sqrt{2}}{\sigma} \quad \begin{array}{l} T > 3\gamma \text{ (no pulse overlap at output)} \\ \text{for } \sigma \ll \gamma \text{ (output pulse width} \gg \text{input pulse width)} \\ h_{\text{det.-filt.}}(t) = h_{j_{\text{out}}}(-t) \text{ (matched filter equalizer).} \end{array} \quad (40)$$

Obviously, the guide acts linearly in power if the output pulses don't overlap, but (40) shows that the output pulses take on a well-defined shape (i.e., the deviations from the ensemble average are negligible) in spite of the random mode coupling. For cases where the output pulses overlap, the conditions for the deviations from the ensemble average to be negligible are more stringent, i.e., the signal-to-noise ratio (39) is less than the special case (40).

Summarizing, we started with a mode-locked laser putting out pulses which were much narrower than the spacing between them and on-off modulated. This optical field excited a guide with random mode coupling. We modeled the transfer function relating the input field complex envelope to the complex amplitude of a particular mode at the output as having specific properties (26) associated with a complex Gaussian random process. This model was justified only in the sense that it was consistent with available but scanty analytical results on guides with random coupling. We showed that the ensemble average output power in the mode under consideration looked like a linearly filtered version of the input power. That is, the average output power was a linear sum of pulses which could in general overlap. Thus, on the average, the guide looked linear in power for digital communication applications. We showed that the deviations in a particular guide from this ensemble average linearity behavior would be negligible provided the input pulses were very narrow compared to the width of the guide average power impulse response. How narrow the input pulses had to be depended upon how much equalization was required to separate the output pulses.†

It is clear that, since the detector adds the powers in each output mode, the total power will be a linear sum of pulses if the individual mode powers are. In addition, we may suspect that the deviations from the ensemble average in each mode may add randomly while the average powers add systematically. Thus, some improvement in the signal to "noise" ratio may accrue from this spatial diversity.

An interpretation of what is happening to make the deviations negligible is the following: Since the guide average power impulse response for output mode j, $s_j(t)$, is the Fourier transform of the two-frequency correlation function $S_j(\omega)$ defined in (26), we can interpret the reciprocal of the width of $s_j(t)$ as the bandwidth difference over which the guide transfer function between the input field and the output mode j becomes uncorrelated. When we use narrow input pulses compared to the width of the average power impulse response, we use a lot of bandwidth compared to this correlation bandwidth and

† Remember that the output pulse shape becomes independent of the input pulse shape as the input pulses get narrow.

thus obtain frequency diversity. As the input pulses become very narrow, the output pulses become fixed in shape equal to $s_j(t)$, but the diversity keeps increasing, resulting in averaging out of the deviations from one guide to the next.

V. CASE IV: INCOHERENT SOURCE, MULTIMODE FIBER WITH MODE COUPLING

In this example, we consider an incoherent intensity-modulated source exciting a multimode fiber with random mode coupling. We assume that material dispersion is negligible. (Since we shall be considering a wideband source, we may question the physical reality of neglecting material dispersion. On the other hand, the qualitative results to follow may provide insight into more general cases.) To simplify what will prove to be a somewhat complicated analysis, we shall consider the response in a particular guide mode, j, at the output, to the field in a particular mode, k, at the guide input. Extension to consideration of the total input field and the total response should be straightforward, using the techniques outlined below.

The input field complex amplitude in mode k is of the form

$$\epsilon_{k_{\text{in}}}(t) = \sqrt{m(t)}c(t), \tag{41}$$

where $m(t)$ is the modulation and $c(t)$ is the optical incoherent carrier which satisfies

$$\langle c(t)c^*(u)\rangle = R_c(t-u) \tag{42a}$$

$$\langle c(t)c^*(u)c(t')c^*(u')\rangle = R_c(t-u)R_c(t'-u') + R_c(t-u')R_c(t'-u). \tag{42b}$$

The complex amplitude in mode j at the output due to the input field in mode k is given by

$$\epsilon_{j_{\text{out}}}(t) = \int \epsilon_{k_{\text{in}}}(t')h_{jk}(t-t')dt', \tag{43}$$

where the impulse response coupling mode k to j is assumed to satisfy the same statistics as were outlined in the first paragraphs of Section IV, i.e.,

$$\begin{aligned}
&(i)\ \ H_{jk}(\omega) = \mathfrak{F}\{h_{jk}(t)\}\\
&(ii)\ \ \langle H_{jk}(\omega)H_{jk}^*(\omega+\sigma)\rangle = S_{jk}(\sigma)\\
&(iii)\ \ \langle H_{jk}(\omega)H_{jk}^*(\omega+\sigma)H_{jk}(\omega+\sigma')H_{jk}^*(\omega+\sigma'')\rangle\\
&\qquad = S_{jk}(\sigma)S_{jk}(\sigma''-\sigma') + S_{jk}(\sigma'')S_{jk}(\sigma-\sigma')\\
&(iv)\ \ \langle h_{jk}(t)h_{jk}^*(t')\rangle = s_{jk}(t)\delta(t-t'),
\end{aligned} \tag{44}$$

where

$$s_{jk}(t) = \mathfrak{F}^{-1}\{S_{jk}(\omega)\}.$$

The average input power (averaging over the source fluctuations) is

$$\langle p_{k_{\text{in}}}(t)\rangle = m(t)R_c(0) = [\sum a_k h_{\text{in}}(t - kT)]R_c(0), \tag{45}$$

i.e., a sequence of on-off modulated pulses where $a_k = 0$ or 1, $T =$ pulse spacing.

The output power is given by

$$p_{j_{\text{out}}}(t) = \iint \sqrt{m(t')}\sqrt{m(t'')}c(t')c^*(t'')h_{jk}(t - t')h_{jk}^*(t - t'')dt'dt''. \tag{46}$$

If we average over the source fluctuations and the guide statistics we obtain, using (44) and (6)

$$\begin{aligned}\langle p_{j_{\text{out}}}(t)\rangle &= \int m(t')R_c(t' - t'')\langle h_{jk}(t - t')h_{jk}^*(t - t'')\rangle dt''dt' \\ &= \int m(t')R_c(0)s_{jk}(t - t')dt' = \sum a_k h_{j_{\text{out}}}(t - kT),\end{aligned} \tag{47}$$

where

$$h_{j_{\text{out}}}(t) = R_c(0)\int h_{\text{in}}(t')s_{jk}(t - t')dt'.$$

Thus, on the average, the output power looks like a linearly filtered version of the input power with impulse response $s_{jk}(t)$.

This average output power will produce an average detected-equalized voltage which is also a linearly filtered version of the input power. We next show that, if the optical bandwidth of the incoherent source is sufficiently large, then the deviations of the detected equalized voltage from its average, because of fluctuations in the source and deviations of the impulse response of a particular guide from the ensemble average, can be neglected.

As before, the detected-equalized voltage is given by

$$v_{\text{out}}(t) = z\int p_{\text{out}}(t')h_{\text{det.-filt.}}(t - t')dt', \tag{48}$$

where z is an arbitrary constant.

The mean detected-equalized current and the mean square deviation from the mean are given by

$$\langle v_{\text{out}}(t)\rangle = \sum a_k h_{v_{\text{out}}}(t - kT),$$

where

$$h_{v_{\text{out}}}(t) = \int h_{j_{\text{out}}}(t')h_{\text{det.-filt.}}(t-t')dt'$$

$$\sigma_v^2(t) = \langle v_{\text{out}}^2(t)\rangle - \langle v_{\text{out}}(t)\rangle^2 = \int \langle p_{j_{\text{out}}}(t')p_{j_{\text{out}}}(t'')\rangle \times h_{\text{det.-filt.}}(t-t')h_{\text{det.-filt.}}(t-t'')dt'dt'' - \langle v_{\text{out}}(t)\rangle^2 \quad (49)$$

To calculate the mean square deviation we need the correlation function of $p_{j_{\text{out}}}(t)$. Using (6), (42b), and (44) we obtain the following complicated expression:

$$\langle p_{j_{\text{out}}}(t')p_{j_{\text{out}}}(t'')\rangle = \langle p_{j_{\text{out}}}(t')\rangle\langle p_{j_{\text{out}}}(t'')\rangle\left[1 + \frac{|R_c(t''-t')|^2}{R_c^2(0)}\right] + \int [m(\tau)]^4 s_{jk}(t'-\tau)s_{jk}(t''-\tau)d\tau \int |R_c(u)|^2 du + \int [m(\tau)]^2[m(t''-t'-\tau)]^2|s_{jk}(t'-\tau)|^2 d\tau \times \int |R_c(u)|^2 du. \quad (50)$$

In order to obtain numerical results we must make some assumptions as to the shapes of $s_{jk}(t)$, $h_{\text{in}}(t)$, and $R_c(t)$. We shall assume the following:

$$\begin{aligned} h_{\text{in}}(t) &= \exp -t^2/2\sigma^2 && \text{(Gaussian-shaped input pulses)} \\ \mathfrak{F}\{R_c(t)\} &= \exp -\omega^2/2B_s^2 && \text{(Gaussian-shaped source spectrum)} \\ s_{jk}(t) &= \exp -t^2/2\gamma^2 && \text{(Gaussian-shaped power impulse response—appropriate for long guides).} \end{aligned} \quad (51)$$

Using (51) in (50) we obtain the expressions

$$\langle p_{j_{\text{out}}}(t)\rangle = \sum a_k e^{[-(t-kT)^2/2(\sigma^2+\gamma^2)]}\frac{\sigma\gamma B_s}{\sqrt{\sigma^2+\gamma^2}}$$

$$\begin{aligned} &\langle p_{j_{\text{out}}}(t')p_{j_{\text{out}}}(t'')\rangle \\ &= \langle p_{j_{\text{out}}}(t')\rangle\langle p_{j_{\text{out}}}(t'')\rangle\{1 + \exp[-(t''-t')^2B_s^2]\} \\ &\quad + \frac{B_s\sigma\gamma}{2\sqrt{\sigma^2+\gamma^2}}\sum\sum a_k a_j\{e^{[-(t''-t')^2/4\gamma^2]}e^{-[(k-j)T]^2/4\sigma^2} \\ &\quad \times e^{[-1/\sigma^2+\gamma^2]\{t'+[(t''-t')/2]-(k+j)(T/2)\}^2} + e^{-[(k-j)T+(t''-t')]^2/4\sigma^2} \\ &\quad \times e^{[-1/\sigma^2+\gamma^2]\{t'+[(t''-t')/2]-(k+j)(T/2)\}^2}\}. \end{aligned} \quad (52)$$

Equation (52) is still fairly complicated, but we can make some general observations. The mean output power is proportional to the source bandwidth B_s, and therefore the mean output power squared will be proportional to B_s^2. Looking at the mean squared output power, we see that the portions involving the double sum are proportional to B_s. Those terms will become negligible compared to the mean output power squared as B_s gets large. The term $\exp -(t'' - t')^2 B_s^2$ can be approximated by an impulse of area $\sqrt{2\pi}/B_s$. Thus, this term is also proportional to B_s^{-1} and will be negligible for large B_s. Thus, for large B_s we can approximate the mean square power by the mean power squared. Equivalently, $\sigma_v^2(t)$ will become negligible compared to $\langle v_{\text{out}}(t) \rangle$ as the source bandwidth becomes very large. Just how large the source bandwidth has to be depends upon how much overlap there is to average output pulses and therefore upon how much equalization has to be done. In the simple case where the output pulses do not overlap and where the equalizer is matched in shape to the output pulses, we obtain

$$\frac{S}{N} = \frac{\langle v_{\text{out}}(t) \rangle^2}{\sigma_v^2(t)} \cong \frac{\sqrt{2} B_s}{\dfrac{1}{\sqrt{\gamma^2 + \sigma^2}} + \dfrac{\sqrt{\gamma^2 + \sigma^2}}{\sigma\sqrt{2\gamma^2 + \sigma^2}} + \dfrac{\sqrt{\gamma^2 + \sigma^2}}{\gamma\sqrt{2\sigma^2 + \gamma^2}}} \tag{53}$$

for

$$T > 3\sqrt{\sigma^2 + \gamma^2} = 1.5 \times \text{output pulse width}$$
$$h_{\text{det.-filt.}}(t) = h_{j_{\text{out}}}(-t) \text{ (matched filter).}$$

From (53) we see that, in this special case, the ratio of the mean detected equalized (filtered) voltage squared to the mean squared deviation, because of source fluctuations and guide random coupling, will be greater than 1000 if the optical source bandwidth is more than 1000 times the reciprocals of both the input pulse duration and the average power impulse response duration.

Typically, the source bandwidth is more than 10^{13} radians per seconds. Thus, the deviations are negligible for input and output pulse widths larger than 10^{-10} seconds. Of course, more bandwidth is required to make the deviations negligible when there is considerable pulse overlap.

An interpretation is similar to previous cases. The requirement that the source bandwidth be large compared to the reciprocal of the duration of the input pulses allows averaging out of the fluctuations in the source. The requirement that the source bandwidth be large compared to the reciprocal of the guide average power impulse response duration gives the frequency diversity that averages out the deviations between guides resulting from random mode coupling.

VI. CONCLUSIONS

We conclude that there are a number of interesting circumstances in which a fiber system, normally linear in voltage, can be considered linear in power for digital communication purposes. The input power consisting of a sum of on-off modulated pulses produces an output power (and thus a detected current) which is also a sequence of on-off modulated pulses, possibly overlapping. The output pulse shape is well-defined under the conditions described above, which usually amount to using enough optical bandwidth to have sufficient frequency diversity to average out source fluctuations and/or random mode coupling differences between guides. This power linearity allows baseband equalization (with the usual noise penalties) to allow use of the guide at higher bit rates than would be associated with the criterion that the output pulses must not overlap. Many assumptions made above may not be completely applicable to particular guides, but it is hoped that, qualitatively, some insight as to when power linearity may occur will be derived from these results.

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